

**Series for Zonal Harmonics.** With  $\mu = \cos \theta$ , the zonal harmonic functions are

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu,$$

$$P_2(\mu) = \frac{1}{2} (3\mu^2 - 1),$$

$$P_3(\mu) = \frac{\mu}{2} (5\mu^2 - 3),$$

$$P_4(\mu) = \frac{1}{8} (35\mu^4 - 30\mu^2 + 3),$$

$$P_5(\mu) = \frac{\mu}{8} (63\mu^4 - 70\mu^2 + 15),$$

$$P_6(\mu) = \frac{1}{16} (231\mu^6 - 315\mu^4 + 105\mu^2 - 5),$$

$$P_7(\mu) = \frac{\mu}{16} (429\mu^6 - 693\mu^4 + 315\mu^2 - 35),$$

$$P_8(\mu) = \frac{1}{128} (6435\mu^8 - 12012\mu^6 + 6930\mu^4 - 1260\mu^2 + 35),$$

and, in general,

$$P_m(\mu) = \frac{(2m-1)(2m-3)\cdots 1}{m!} \left[ \mu^m - \frac{m(m-1)}{2(2m-1)} \mu^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4(2m-1)(2m-3)} \mu^{m-4} - \cdots \right].$$

The series terminates with the term in  $\mu$ , if  $m$  is odd, and with the term independent of  $\mu$ , if  $m$  is even.

$$\text{For large values of } m, P_m(\cos \theta) \cong \left( \frac{2}{m\pi \sin \theta} \right)^{1/2} \sin \left\{ \left( m + \frac{1}{2} \right) \theta + \frac{\pi}{4} \right\}.$$

The recursion formula  $(m+1)P_{m+1}(\mu) = (2m+1)\mu P_m(\mu) - mP_{m-1}(\mu)$  may be useful.

$$\text{Differential Coefficients, } \frac{d}{d\mu} [P_m(\mu)] = P_m'(\mu).$$

$$P_0'(\mu) = 0, \quad P_1'(\mu) = 1, \quad P_2'(\mu) = 3\mu.$$

$$P_3'(\mu) = \frac{1}{2} (3.5\mu^2 - 1.3),$$

$$P_4'(\mu) = \frac{1}{2} (5 \cdot 7 \mu^3 - 3 \cdot 5 \mu),$$

$$P_5'(\mu) = \frac{1}{2 \cdot 4} (5 \cdot 7 \cdot 9 \mu^4 - 2 \cdot 3 \cdot 5 \cdot 7 \mu^2 + 1 \cdot 3 \cdot 5),$$

$$P_6'(\mu) = \frac{1}{2 \cdot 4} (7 \cdot 9 \cdot 11 \mu^5 - 2 \cdot 5 \cdot 7 \cdot 9 \mu^3 + 3 \cdot 5 \cdot 7 \mu),$$

$$P_7'(\mu) = \frac{1}{2 \cdot 4 \cdot 6} (7 \cdot 9 \cdot 11 \cdot 13 \mu^6 - 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \mu^4 + 3 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \mu^2 - 1 \cdot 3 \cdot 5 \cdot 7),$$

with the general relation,

$$(\mu^2 - 1)P_m'(\mu) = m\mu P_m(\mu) - mP_{m-1}(\mu).$$

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AUXILIARY TABLE 4. VALUES OF DIFFERENTIAL COEFFICIENTS OF ZONAL HARMONICS

| $\nu$ | $P_3'(\nu)$ | $\Delta'$ | $P_6'(\nu)$ | $\Delta'$ | $\Delta''$ | $P_7'(\nu)$ | $\Delta'$ | $\Delta''$ | $\nu$ |
|-------|-------------|-----------|-------------|-----------|------------|-------------|-----------|------------|-------|
| 0     | -1.5000     |           | 1.8750      |           |            | -2.1873     |           |            | 0     |
| 0.01  | 1.4992      | + 8       | 1.8725      | - 25      | -55        | 2.1816      | 57        | 120        | 0.01  |
| .02   | 1.4970      | + 22      | 1.8645      | - 80      | -51        | 2.1639      | 177       | 117        | .02   |
| .03   | 1.4932      | + 38      | 1.8514      | - 131     | -52        | 2.1345      | 294       | 115        | .03   |
| .04   | 1.4880      | + 52      | 1.8331      | - 183     | -52        | 2.0936      | 409       | 115        | .04   |
| 0.05  | -1.4812     | + 68      | 1.8096      | - 235     | -51        | -2.0412     | 524       | 111        | 0.05  |
| .06   | 1.4730      | + 82      | 1.7810      | - 286     | -51        | 1.9777      | 635       | 110        | .06   |
| .07   | 1.4632      | + 98      | 1.7473      | - 337     | -50        | 1.9032      | 745       | 104        | .07   |
| .08   | 1.4520      | +112      | 1.7086      | - 387     | -50        | 1.8183      | 849       | 102        | .08   |
| .09   | 1.4392      | +128      | 1.6649      | - 437     | -48        | 1.7232      | 951       | 98         | .09   |
| 0.10  | -1.4250     | +142      | 1.6164      | - 485     | -48        | -1.6183     | 1049      | 92         | 0.10  |
| .11   | 1.4092      | +158      | 1.5631      | - 533     | -47        | 1.5042      | 1141      | 88         | .11   |
| .12   | 1.3920      | +172      | 1.5051      | - 580     | -45        | 1.3813      | 1229      | 81         | .12   |
| .13   | 1.3732      | +188      | 1.4426      | - 625     | -45        | 1.2503      | 1310      | 76         | .13   |
| .14   | 1.3530      | +202      | 1.3756      | - 670     | -43        | 1.1117      | 1386      | 70         | .14   |
| 0.15  | -1.3312     | +218      | 1.3043      | - 713     | -42        | -0.9661     | 1456      | 62         | 0.15  |
| .16   | 1.3080      | +232      | 1.2288      | - 755     | -40        | .8143       | 1518      | 55         | .16   |
| .17   | 1.2832      | +248      | 1.1493      | - 795     | -40        | .6570       | 1573      | 48         | .17   |
| .18   | 1.2570      | +262      | 1.0658      | - 835     | -36        | .4949       | 1621      | 41         | .18   |
| .19   | 1.2292      | +278      | 0.9787      | - 871     | -36        | .3287       | 1662      | 30         | .19   |
| 0.20  | -1.2000     | +292      | 0.8880      | - 907     | -34        | -0.1595     | 1692      | 25         | 0.20  |
| .21   | 1.1692      | +308      | .7939       | - 941     | -31        | +0.0122     | 1717      | 14         | .21   |
| .22   | 1.1370      | +322      | .6967       | - 972     | -29        | .1853       | 1731      | + 5        | .22   |
| .23   | 1.1032      | +338      | .5966       | -1001     | -29        | .3589       | 1736      | - 4        | .23   |
| .24   | 1.0680      | +352      | .4936       | -1030     | -24        | .5321       | 1732      | -15        | .24   |
| 0.25  | -1.0312     | +368      | 0.3882      | -1054     |            | 0.7038      | 1717      |            | 0.25  |

AUXILIARY TABLE 4. DIFFERENTIAL COEFFICIENTS OF ZONAL HARMONICS (*Continued*)

| $\nu$ | $P_3'(\nu)$ | $\Delta'$ | $P_6'(\nu)$ | $\Delta'$ | $\Delta''$ | $P_7'(\nu)$ | $\Delta'$ | $\Delta''$ | $\nu$ |
|-------|-------------|-----------|-------------|-----------|------------|-------------|-----------|------------|-------|
| 0.25  | -1.0312     |           | 0.3882      |           | -23        | 0.7038      |           | -20        | 0.25  |
|       |             | 382       |             | -1077     |            |             | 1697      |            |       |
| .26   | 0.9930      |           | .2805       |           | -21        | 0.8735      |           | -32        | .26   |
|       |             | 398       |             | -1098     |            |             | 1665      |            |       |
| .27   | .9532       |           | .1707       |           | -18        | 1.0400      |           | -41        | .27   |
|       |             | 412       |             | -1116     |            |             | 1624      |            |       |
| .28   | .9120       |           | +0.0591     |           | -16        | 1.2024      |           | -51        | .28   |
|       |             | 428       |             | -1132     |            |             | 1573      |            |       |
| .29   | .8692       |           | -0.0541     |           | -13        | 1.3597      |           | -60        | .29   |
|       |             | 442       |             | -1145     |            |             | 1513      |            |       |
| 0.30  | -0.8250     |           | .1686       |           | -9         | 1.5110      |           | -71        | 0.30  |
|       |             | 458       |             | -1154     |            |             | 1442      |            |       |
| .31   | .7792       |           | .2840       |           | -7         | 1.6552      |           | -79        | .31   |
|       |             | 472       |             | -1161     |            |             | 1363      |            |       |
| .32   | .7320       |           | .4001       |           | -4         | 1.7915      |           | -91        | .32   |
|       |             | 488       |             | -1165     |            |             | 1272      |            |       |
| .33   | .6832       |           | .5166       |           | -2         | 1.9187      |           | -98        | .33   |
|       |             | 502       |             | -1167     |            |             | 1174      |            |       |
| .34   | .6330       |           | .6333       |           | +2         | 2.0361      |           | -106       | .34   |
|       |             | 518       |             | -1165     |            |             | 1068      |            |       |
| 0.35  | -0.5812     |           | -0.7498     |           | 6          | 2.1429      |           | -116       | 0.35  |
|       |             | 532       |             | -1159     |            |             | 952       |            |       |
| .36   | .5280       |           | .8657       |           | 9          | 2.2381      |           | -124       | .36   |
|       |             | 548       |             | -1150     |            |             | 828       |            |       |
| .37   | .4732       |           | -0.9807     |           | 12         | 2.3209      |           | -132       | .37   |
|       |             | 562       |             | -1138     |            |             | 696       |            |       |
| .38   | .4170       |           | -1.0945     |           | 15         | 2.3905      |           | -139       | .38   |
|       |             | 578       |             | -1123     |            |             | 557       |            |       |
| .39   | .3592       |           | 1.2068      |           | 21         | 2.4462      |           | -147       | .39   |
|       |             | 592       |             | -1102     |            |             | 410       |            |       |
| 0.40  | -0.3000     |           | -1.3170     |           | 22         | 2.4872      |           | -154       | 0.40  |
|       |             | 608       |             | -1080     |            |             | 256       |            |       |
| .41   | .2392       |           | 1.4250      |           | 27         | 2.5128      |           | -159       | .41   |
|       |             | 622       |             | -1053     |            |             | + 97      |            |       |
| .42   | .1770       |           | 1.5303      |           | 31         | 2.5225      |           | -165       | .42   |
|       |             | 638       |             | -1022     |            |             | - 68      |            |       |
| .43   | .1132       |           | 1.6325      |           | 35         | 2.5157      |           | -170       | .43   |
|       |             | 652       |             | - 987     |            |             | - 238     |            |       |
| .44   | -0.0480     |           | 1.7312      |           | 39         | 2.4919      |           | -173       | .44   |
|       |             | 668       |             | - 948     |            |             | - 411     |            |       |
| 0.45  | +0.0188     |           | -1.8260     |           | 43         | 2.4508      |           | -178       | 0.45  |
|       |             | 682       |             | - 905     |            |             | - 589     |            |       |
| .46   | .0870       |           | 1.9165      |           | 48         | 2.3919      |           | -179       | .46   |
|       |             | 698       |             | - 857     |            |             | - 768     |            |       |
| .47   | .1568       |           | 2.0022      |           | 51         | 2.3151      |           | -182       | .47   |
|       |             | 712       |             | - 806     |            |             | - 950     |            |       |
| .48   | .2280       |           | 2.0828      |           | 57         | 2.2201      |           | -181       | .48   |
|       |             | 728       |             | - 749     |            |             | -1131     |            |       |
| .49   | .3008       |           | 2.1577      |           | 60         | 2.1070      |           | -183       | .49   |
|       |             | 742       |             | - 689     |            |             | -1314     |            |       |
| 0.50  | +0.3750     |           | -2.2266     |           | 67         | 1.9756      |           | -178       | 0.50  |

AUXILIARY TABLE 4. DIFFERENTIAL COEFFICIENTS OF ZONAL HARMONICS (Continued)

| $\nu$ | $P_3'(\nu)$ | $\Delta'$ | $P_5'(\nu)$ | $\Delta'$ | $\Delta''$ | $P_7'(\nu)$ | $\Delta'$ | $\Delta''$ | $\nu$ |
|-------|-------------|-----------|-------------|-----------|------------|-------------|-----------|------------|-------|
| 0.50  | 0.3750      |           | -2.2266     |           | 67         | 1.9756      |           | -178       | 0.50  |
|       |             | 758       |             | -622      |            |             | -1492     |            |       |
| .51   | .4508       |           | 2.2888      |           | 70         | 1.8264      |           | -177       | .51   |
|       |             | 772       |             | -552      |            |             | -1669     |            |       |
| .52   | .5280       |           | 2.3440      |           | 75         | 1.6595      |           | -173       | .52   |
|       |             | 788       |             | -477      |            |             | -1842     |            |       |
| .53   | .6068       |           | 2.3917      |           | 80         | 1.4753      |           | -167       | .53   |
|       |             | 802       |             | -397      |            |             | -2009     |            |       |
| .54   | .6870       |           | 2.4314      |           | 85         | 1.2744      |           | -161       | .54   |
|       |             | 818       |             | -312      |            |             | -2170     |            |       |
| 0.55  | 0.7688      |           | -2.4626     |           | 91         | 1.0574      |           | -153       | 0.55  |
|       |             | 832       |             | -221      |            |             | -2323     |            |       |
| .56   | .8520       |           | 2.4847      |           | 96         | 0.8251      |           | -142       | .56   |
|       |             | 848       |             | -125      |            |             | -2465     |            |       |
| .57   | 0.9368      |           | 2.4972      |           | 101        | 0.5786      |           | -133       | .57   |
|       |             | 862       |             | - 24      |            |             | -2598     |            |       |
| .58   | 1.0230      |           | 2.4996      |           | 106        | 0.3188      |           | -117       | .58   |
|       |             | 878       |             | + 82      |            |             | -2715     |            |       |
| .59   | 1.1108      |           | 2.4914      |           | 112        | +0.0473     |           | -105       | .59   |
|       |             | 892       |             | 194       |            |             | -2820     |            |       |
| 0.60  | 1.2000      |           | -2.4720     |           | 118        | -0.2347     |           | - 88       | 0.60  |
|       |             | 908       |             | 312       |            |             | -2908     |            |       |
| .61   | 1.2908      |           | 2.4408      |           | 123        | 0.5255      |           | - 69       | .61   |
|       |             | 922       |             | 435       |            |             | -2977     |            |       |
| .62   | 1.3830      |           | 2.3973      |           | 129        | 0.8232      |           | - 51       | .62   |
|       |             | 938       |             | 564       |            |             | -3028     |            |       |
| .63   | 1.4768      |           | 2.3409      |           | 136        | 1.1260      |           | - 21       | .63   |
|       |             | 952       |             | 700       |            |             | -3049     |            |       |
| .64   | 1.5720      |           | 2.2709      |           | 140        | 1.4309      |           | - 4        | .64   |
|       |             | 968       |             | 840       |            |             | -3053     |            |       |
| 0.65  | 1.6688      |           | -2.1869     |           | 147        | -1.7362     |           | + 26       | 0.65  |
|       |             | 982       |             | 987       |            |             | -3027     |            |       |
| .66   | 1.7670      |           | 2.0882      |           | 154        | 2.0389      |           | 54         | .66   |
|       |             | 998       |             | 1141      |            |             | -2973     |            |       |
| .67   | 1.8668      |           | 1.9741      |           | 160        | 2.3362      |           | 87         | .67   |
|       |             | 1012      |             | 1301      |            |             | -2886     |            |       |
| .68   | 1.9680      |           | 1.8440      |           | 165        | 2.6248      |           | 120        | .68   |
|       |             | 1028      |             | 1466      |            |             | -2766     |            |       |
| .69   | 2.0608      |           | 1.6974      |           | 172        | 2.9014      |           | 156        | .69   |
|       |             | 1042      |             | 1638      |            |             | -2610     |            |       |
| 0.70  | 2.1750      |           | -1.5336     |           | 180        | -3.1624     |           | 200        | 0.70  |
|       |             | 1058      |             | 1818      |            |             | -2410     |            |       |
| .71   | 2.2808      |           | 1.3518      |           | 186        | -3.4034     |           | 237        | .71   |
|       |             | 1072      |             | 2004      |            |             | -2173     |            |       |
| .72   | 2.3880      |           | 1.1514      |           | 193        | 3.6207      |           | 285        | .72   |
|       |             | 1088      |             | 2197      |            |             | -1888     |            |       |
| .73   | 2.4968      |           | 0.9317      |           | 198        | 3.8095      |           | 333        | .73   |
|       |             | 1102      |             | 2395      |            |             | -1555     |            |       |
| .74   | 2.6070      |           | 0.6922      |           | 206        | 3.9650      |           | 383        | .74   |
|       |             | 1118      |             | 2601      |            |             | -1172     |            |       |
| 0.75  | 2.7188      |           | -0.4321     |           | 214        | -4.0822     |           | 439        | 0.75  |

AUXILIARY TABLE 4. DIFFERENTIAL COEFFICIENTS OF ZONAL HARMONICS (*Concluded*)

| $\nu$ | $P_3'(\nu)$ | $\Delta'$ | $P_5'(\nu)$ | $\Delta'$ | $\Delta''$ | $P_7'(\nu)$ | $\Delta'$ | $\Delta''$ | $\Delta'''$ | $\nu$ |
|-------|-------------|-----------|-------------|-----------|------------|-------------|-----------|------------|-------------|-------|
| 0.75  | 2.7188      |           | -0.4321     |           | 214        | -4.0822     |           | 439        |             | 0.75  |
|       |             | 1132      |             | 2815      |            |             | -733      |            | 57          |       |
| .76   | 2.8320      |           | -0.1506     |           | 220        | 4.1555      |           | 496        |             | .76   |
|       |             | 1148      |             | 3035      |            |             | -237      |            | 60          |       |
| .77   | 2.9468      |           | +0.1529     |           | 228        | 4.1792      |           | 556        |             | .77   |
|       |             | 1162      |             | 3263      |            |             | +319      |            | 66          |       |
| .78   | 3.0630      |           | 0.4792      |           | 234        | 4.1473      |           | 622        |             | .78   |
|       |             | 1178      |             | 3497      |            |             | 941       |            | 66          |       |
| .79   | 3.1808      |           | 0.8289      |           | 244        | 4.0532      |           | 688        |             | .79   |
|       |             | 1192      |             | 3741      |            |             | 1629      |            | 76          |       |
| 0.80  | 3.3000      |           | 1.2030      |           | 249        | -3.8903     |           | 764        |             | 0.80  |
|       |             | 1208      |             | 3990      |            |             | 2393      |            | 71          |       |
| .81   | 3.4208      |           | 1.6020      |           | 258        | 3.6510      |           | 835        |             | .81   |
|       |             | 1222      |             | 4248      |            |             | 3228      |            | 83          |       |
| .82   | 3.5430      |           | 2.0268      |           | 265        | 3.3282      |           | 918        |             | .82   |
|       |             | 1238      |             | 4513      |            |             | 4146      |            | 89          |       |
| .83   | 3.6668      |           | 2.4781      |           | 273        | 2.9136      |           | 1002       |             | .83   |
|       |             | 1252      |             | 4786      |            |             | 5148      |            | 82          |       |
| .84   | 3.7920      |           | 2.9567      |           | 281        | 2.3988      |           | 1084       |             | .84   |
|       |             | 1268      |             | 5067      |            |             | 6232      |            | 99          |       |
| 0.85  | 3.9188      |           | 3.4634      |           | 289        | -1.7756     |           | 1183       |             | 0.85  |
|       |             | 1282      |             | 5356      |            |             | 7415      |            | 93          |       |
| .86   | 4.0470      |           | 3.9090      |           | 297        | 1.0341      |           | 1276       |             | .86   |
|       |             | 1298      |             | 5653      |            |             | 8691      |            | 101         |       |
| .87   | 4.1768      |           | 4.5643      |           | 304        | -0.1650     |           | 1377       |             | .87   |
|       |             | 1312      |             | 5957      |            |             | 10068     |            | 110         |       |
| .88   | 4.3080      |           | 5.1600      |           | 315        | +0.8418     |           | 1487       |             | .88   |
|       |             | 1328      |             | 6272      |            |             | 11555     |            | 108         |       |
| .89   | 4.4408      |           | 5.7872      |           | 320        | 1.9963      |           | 1595       |             | .89   |
|       |             | 1342      |             | 6592      |            |             | 13150     |            |             |       |
| 0.90  | 4.5750      |           | 6.4464      |           | 332        | 3.3113      |           |            |             | 0.90  |
|       |             | 1358      |             | 6924      |            |             |           |            |             |       |
| .91   | 4.7108      |           | 7.1388      |           | 338        | 4.7965      |           |            |             | .91   |
|       |             | 1372      |             | 7262      |            |             |           |            |             |       |
| .92   | 4.8480      |           | 7.8650      |           | 347        | 6.4644      |           |            |             | .92   |
|       |             | 1388      |             | 7609      |            |             |           |            |             |       |
| .93   | 4.9868      |           | 8.6259      |           | 357        | 8.3276      |           |            |             | .93   |
|       |             | 1402      |             | 7966      |            |             |           |            |             |       |
| .94   | 5.1270      |           | 9.4225      |           | 364        | 10.3991     |           |            |             | .94   |
|       |             | 1418      |             | 8330      |            |             |           |            |             |       |
| 0.95  | 5.2688      |           | 10.2555     |           | 375        | 12.6925     |           |            |             | 0.95  |
|       |             | 1432      |             | 8705      |            |             |           |            |             |       |
| .96   | 5.4120      |           | 11.1260     |           | 383        | 15.2218     |           |            |             | .96   |
|       |             | 1448      |             | 9088      |            |             |           |            |             |       |
| .97   | 5.5568      |           | 12.0348     |           | 392        | 18.0016     |           |            |             | .97   |
|       |             | 1462      |             | 9480      |            |             |           |            |             |       |
| .98   | 5.7030      |           | 12.9828     |           | 401        | 21.0474     |           |            |             | .98   |
|       |             | 1478      |             | 9881      |            |             |           |            |             |       |
| 0.99  | 5.8508      |           | 13.9709     |           | 410        | 24.3746     |           |            |             | 0.99  |
|       |             | 1492      |             | 10291     |            |             |           |            |             |       |
| 1.00  | 6.0000      |           | 15.0000     |           |            | 28.0000     |           |            |             | 1.00  |