Resonance In Lossy Tuned Circuits

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10/13/2013

Abstract

Almost all radio receivers make use of at least one tuned circuit that is driven by a voltage source and feeds the voltage that develops across the tuned circuit into a high input impedance gain stage. Typically, the voltage source will be an inductively coupled antenna circuit and the input of the gain stage will be the grid of a vacuum tube in vintage radios or the gate of a field effect transistor in modern day receivers. An often neglected property of such a setup is that while the maximum current in the tuned circuit always occurs at $\omega = 1/\sqrt{LC}$ the frequency where the voltage across the tuned circuit reaches it's maximum deviates from this frequency. As this deviation turns out to be very small for tuned circuits having a reasonably high Q-factor and is therefore neglectable for most receiver designs, it is still one of the keys to an in-depth understanding of the behavior of tuned circuits.

The Driven Lossy Tuned Circuit

We'll base our considerations on a tuned circuit where the losses occurring in the capacitor C can be neglected against the losses occurring in the inductor L. The losses in the inductor are modeled by a loss resistor R_s connected in series with the inductor and the circuit shall be driven by a voltage source $U_0(t)$. This equivalent circuit is depicted in figure 1

Since the series loss resistor R_s is intrinsic to the inductor, it always needs to be connected directly to it and can not be moved to another place in the circuit. In most cases, the voltage source will be a voltage coupled inductively into the coil from a previous stage or the antenna circuit. If so, the voltage source $U_0(t)$ is also intrinsic to the inductor and therefore also needs to be connected directly to it in this equivalent circuit. Hence, the voltage developing across the tuned circuit simply is the voltage $U_C(t)$ that can be measured at the capacitor.

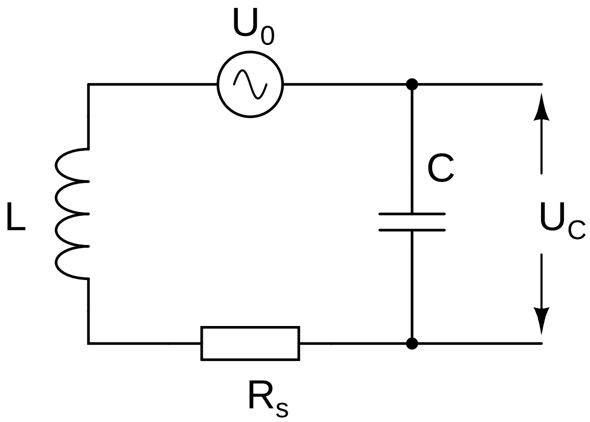

Figure 1: Driven lossy tuned circuit

The Current Response Function

Since we are only interested in the response of the system to a sinusoidal driving voltage $U_0(t)$ we can waive solving the differential equations governing the lossy tuned circuit and use Ohm's law with complex impedances the engineer is most familiar with. The complex driving voltage at frequency ω is assumed to be

$$U_0(t) = \hat{U}_0 e^{j\omega t}$$

with it's amplitude $\hat{U}_0$ being real-valued. The complex current amplitude $\hat{I}$ in the circuit is then given by

$$\hat{I} = \frac{\hat{U}_0}{Z}$$

where

$$Z = R_s + \frac{1}{j\omega C} + j\omega L$$

is the complex impedance of L, C and R_s connected in series. From these equations, one quickly calculates $\hat{I}$ to be

$$\hat{I} = \hat{U}_0 \frac{R_s - j\left(\omega L - \frac{1}{\omega C}\right)}{R_s^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{1}$$

From the above equation, the absolute value of the current amplitude in the circuit turns out to be

$$\boxed{|\hat{I}| = \frac{\hat{U}_0}{\sqrt{R_s^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}} \tag{2}$$

reaching it's maximum at

$$\boxed{\omega_0 = \sqrt{\frac{1}{LC}}}$$

The Capacitor Voltage Response Function

Let us now turn to the voltage U_C that develops across the tuned circuit and that can be found at the capacitor. By virtue of Ohm's law for complex impedances, it's complex amplitude is given by

$$\hat{U}_C = Z_C \hat{I}$$

where

$$Z_C = \frac{1}{j\omega C}$$

is the complex impedance of the capacitor. Using $\hat{I}$ from equation (1) it follows that

$$\hat{U}_C = \hat{U}_0 \frac{1 - \omega^2 LC - j\omega R_s C}{(1 - \omega^2 LC)^2 + (\omega R_s C)^2}$$

and from this expression, the absolute value of the voltage amplitude at the capacitor is easily determined to be

$$|\hat{U}_C| = \frac{\hat{U}_0}{\sqrt{(1 - \omega^2 LC)^2 + (\omega R_s C)^2}}$$

completing the square in the argument of the square root function then finally yields:

$$\boxed{|\hat{U}_C| = \frac{\hat{U}_0}{\sqrt{\left(LC\omega^2 + \frac{R_s^2 C}{2L} - 1\right)^2 - \left(\frac{R_s^2 C}{2L} - 1\right)^2 + 1}}} \tag{3}$$

From this equation it can be seen that the voltage amplitude at the capacitor reaches it's maximum for a driving frequency of

$$\boxed{\omega_C = \sqrt{\frac{1}{LC} - \frac{R_s^2}{2L^2}}} \qquad (4)$$

that deviates from the frequency $\omega_0 = \sqrt{1/LC}$ where the current amplitude $\hat{I}$ in the circuit reaches it's maximum.

Analysis of the Results

For a detailed discussion of the implications of the formulas obtained in the previous sections, it is convenient to replace the series loss resistance R_s by the Q-factor of the tuned circuit. Series loss resistance and Q-factor of a tuned circuit are related by [2]

$$Q = \frac{1}{R_s}\sqrt{\frac{L}{C}}$$

using the above expression along with $\omega_0 = \sqrt{1/LC}$ in equation (4) the driving frequency where the voltage at the capacitor reaches it's maximum can be written as

$$\boxed{\omega_C = \omega_0\sqrt{1 - \frac{1}{2Q^2}}}$$

From the above equation, the relative frequency deviation is given as

$$\frac{\Delta\omega}{\omega_0} = \frac{\omega_0 - \omega_C}{\omega_0} = 1 - \sqrt{1 - \frac{1}{2Q^2}}$$

For reasonably high Q-factors, the frequency deviation is very small. Figure 2 shows the relative frequency deviation for Q-factors ranging from 20 to 80.

It becomes obvious that in most practical applications such small deviations are neglectable, explaining why an analysis of this frequency shift due to losses in the tuned circuit is mostly absent in practically oriented textbooks. It is still worth noting that this frequency shift is a general phenomenon in all harmonic oscillators provided that the damping force (friction) is “ohmic”. See [3] for a general introduction.

Let us now turn to the frequency response curves of the tuned circuit. Using the Q-factor instead of the series loss resistance, the absolute value of the current amplitude $|\hat{I}|$ in the tuned circuit given in equation (2) can be written as

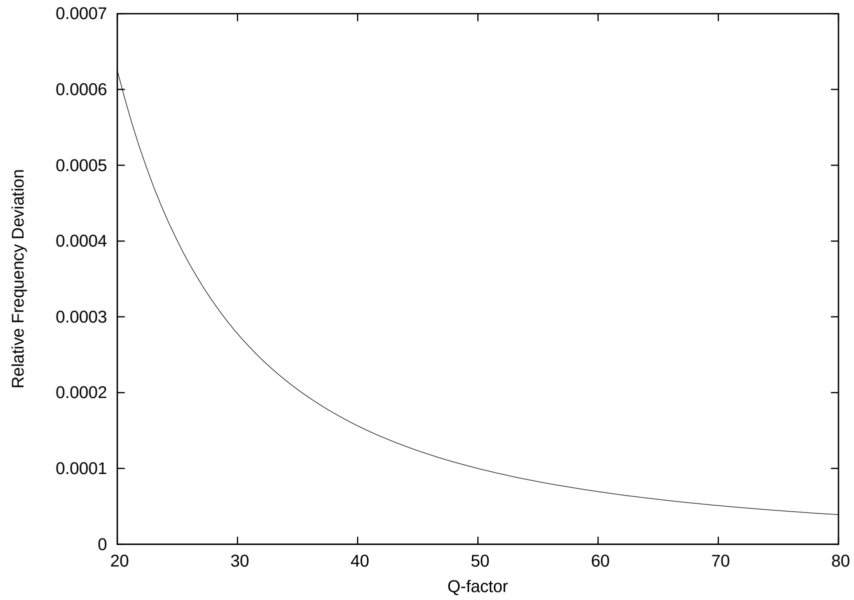

Figure 2: Relative frequency deviation as a function of the Q-factor

$$|\hat{I}| = \frac{\hat{U}_0\sqrt{\frac{C}{L}}}{\sqrt{\frac{1}{Q^2} + \left(\frac{\omega}{\omega_0} - \frac{1}{\frac{\omega}{\omega_0}}\right)^2}}$$

We can now use the maximum current amplitude

$$|\hat{I}|_M = \frac{\hat{U}_0\sqrt{\frac{C}{L}}}{\frac{1}{Q}}$$

occurring at $\omega_0 = \sqrt{1/LC}$ to obtain the normalized current amplitude response function of the tuned circuit:

$$\boxed{\frac{|\hat{I}|}{|\hat{I}|_M} = \frac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{1}{\frac{\omega}{\omega_0}}\right)^2}}}$$

Let us now again turn to the capacitor voltage response function. Replacing the series loss resistance by the Q-factor, we can write equation (3) as:

$$|\hat{U}_C| = \frac{\hat{U}_0}{\sqrt{\left(\left(\frac{\omega}{\omega_0}\right)^2 + \frac{1}{2Q^2} - 1\right)^2 - \left(\frac{1}{2Q^2} - 1\right)^2 + 1}} \tag{5}$$

From which the maximum capacitor voltage amplitude follows to be

$$|\hat{U}_C|_M = \frac{\hat{U}_0}{\sqrt{1 - \left(1 - \frac{1}{2Q^2}\right)^2}}$$

allowing us to state the normalized capacitor voltage response function:

$$\boxed{\frac{|\hat{U}_C|}{|\hat{U}_C|_M} = \frac{1}{\sqrt{\frac{1}{1-\left(1-\frac{1}{2Q^2}\right)^2}\left(\left(\frac{\omega}{\omega_0}\right)^2 - \left(1 - \frac{1}{2Q^2}\right)\right)^2 + 1}}}$$

Note that we have used ω/ω_0 as the normalized frequency in both functions, allowing us to easily create comparison plots of the current and capacitor voltage response function. We'll start with a very low Q-factor of $Q = 2$ producing two quite distinct response curves as shown in figure 3.

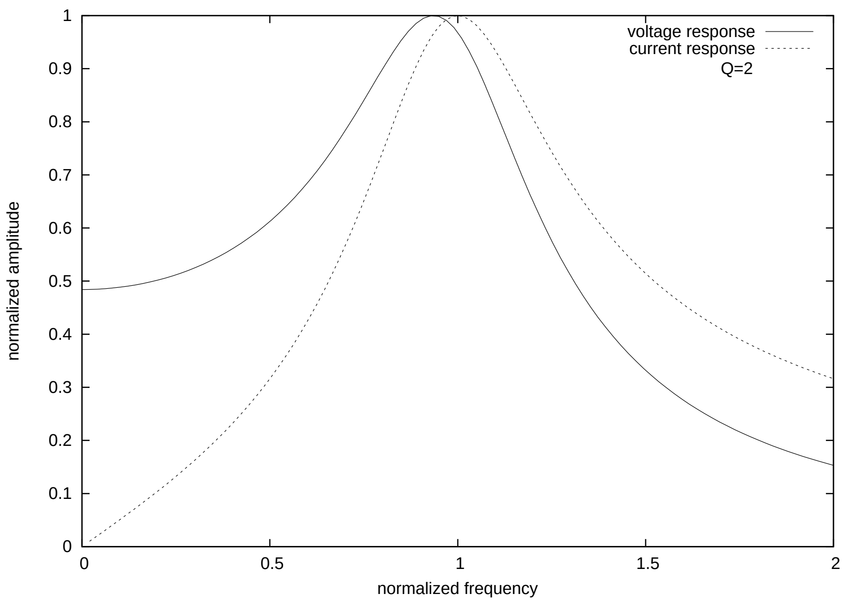

Figure 3: Current and voltage response for Q=2

In most practical circuits, the Q-factor will however be a lot higher then $Q = 2$. As the the Q-factor increases, the normalized current and voltage response functions become a lot more similar to each other. This can for example be seen from the the response functions for $Q = 10$ shown in figure 4.

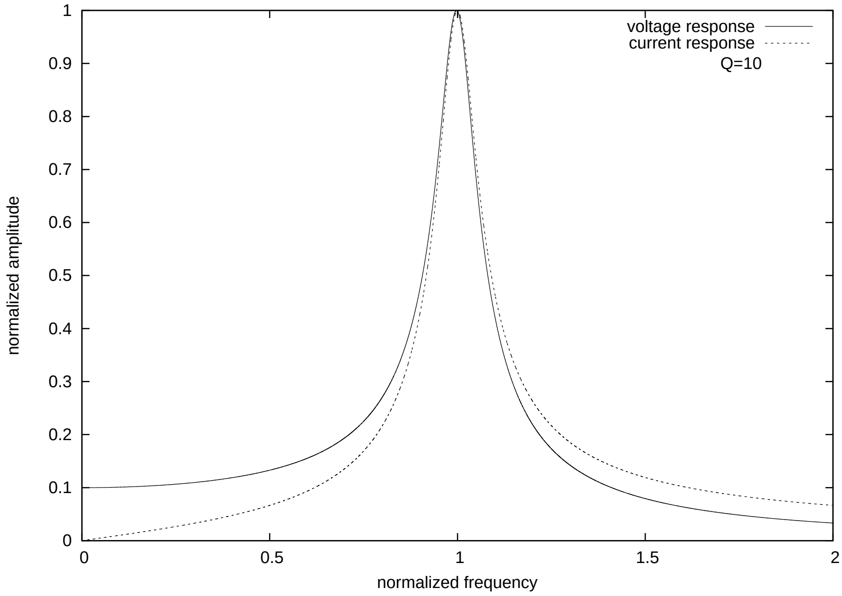

Figure 4: Current and voltage response for Q=10

References

[1] http://www.radiomuseum.org/forumdata/users/133/PDF/RegenerationLC.pdf

[2] http://en.wikipedia.org/wiki/Q_factor

[3] http://en.wikipedia.org/wiki/Harmonic_oscillator